B.M.S COLLEGE FOR WOMEN AUTONOMOUS

BENGALURU – 560004

END SEMESTER EXAMINATION – OCTOBER 2022

M.Sc. in Mathematics – II Semester

Algebra – II

Course Code: MM201T | **QP Code: 21001**
Duration: 3 Hours | **Max marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. (a)Define (i) Nil radical N(A), (ii) Jacobson radical J(A) of a A. Prove that $x \in J(A)$ if and only if $1 - xy$ is a unit in A for all $x \in A$.
   (b) Define extension and contraction of ideals with respect to a ring. If C denote the set of all contracted ideals I of A and E denotes the set of all extended ideals J of B then show that $C = \{I : I^{ec} = I\}$ and $E = \{J : J^{ce} = J\}$.
   (c) Explain the following (i) Operations on ideals. (ii) Prime spectrum of a ring A.

(5+5+4)

2. (a) Show that every abelian group G is a module over the ring of integers.
   (b) State and prove fundamental theorem of Homomorphism on Modules.
   (c) (i) Prove that the kernel of a homomorphism is a sub module.(ii) Prove that the range of a homomorphism is a sub module.

(5+5+4)

3. (a) Define an exact sequence of an A-module. Show that the sequence of A-module and A-linear maps $M' \xrightarrow{u} M \xrightarrow{v} M" \to O$ is exact, if for all A-modules N, the sequence
   $O \to Hom(M",N) \xrightarrow{\bar{v}} Hom(M,N) \xrightarrow{\bar{u}} Hom(M',N)$ is exact.
   (b) Define a simple module. Show that an A-module M is simple if and only if $M \cong \frac{A}{I}$ as for some maximal ideal I of A .

(7+7)

4. (a) Prove that A-module M is of finite length if and only if it is both Noetherian and Artinian.
   (b) Prove that a commutative ring with identity is Noetherian if and only strictly ascending chain of ideals is of finite length.

(7+7)

5. (a) Define an algebraic extension of a field. If L is an algebraic extension of K and K is an algebraic extension of F then prove that L is an algebraic extension of F.

(b) Prove that the elements in an extension K of a field F which are algebraic over F form a subfield of K.

(c) Let $a = \sqrt{2}, b = \sqrt[4]{2}$, where R is an extension of Q. Verify that $(a+b)$ and (ab) are algebraic of degree atmost $(deg\ a)(deg\ b)$.

(5+5+4)

6. (a) Show that a regular pentagon is constructible.

   (b) Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Then prove that there is an extension E of F of degree atmost n! in which $f(x)$ has n roots.

   (c) Determine the splitting field of $x^4 - 2$ over Q.

(5+5+4)

7. (a) Show that the polynomial $f(x) \in F[x]$ has multiple roots if and only if $f(x)$ and $f'(x)$ have a non-trivial common factors.

   (b) Prove that any finite extension of a field of characteristic 0 is simple extension.

   (c) Show that any field of characteristic 0 is perfect field.

(6+4+4)

8. (a) Let $A(F)$ be the collection of all automorphism of a field F. Prove that $A(F)$ is group with respect to the operation known as composite of two functions.

   (b) Let K be a finite extension of a field F of characteristic zero and H be a subgroup of G(K,F). Suppose K_H is the fixed field of H. Then show that

   (i) $[K: K_H] = o(H)$

   (ii) $H = G(K, K_H)$.

(4+10)

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